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Crystallographic symmetries of stochastic webs

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Abstract. In studies of the dynamics of a kicked pendulum, webs along which stochastic diffusion takes place, have been observed to occur at resonance. This paper deals with stochastic webs with crystallographic symmetries. Via an analysis of the (reversing) symmetries of the equations of motion, symmetry properties of stochastic webs are revealed. Furthermore, the threshold for stochastic diffusion is discussed in relation to the (reversing) symmetries of the equations of motion.

1. Introduction

In the last two decades, chaotic motion in nonlinear dynamical systems has been studied extensively. A lot of attention was paid to the two-dimensional standard map that displays chaotic motion if it is perturbed from integrability [1, 2]. In the standard map, the chaotic motion is trapped between KAM-curves if the perturbation from integrability is smaller than some threshold value. This causes a separation of chaotic regions in so-called stochastic layers. Only if the perturbation exceeds the threshold value, the chaotic motion is not trapped in the stochastic layers connect to form a two-dimensional stochastic web. The latter process is called stochastic diffusion. In the standard map, the perturbation threshold for stochastic diffusion is related to the break-up of the last KAM-curve.

A model in which stochastic diffusion occurs, different from the standard map, was introduced by Zaslavsky *et al* in a study of the acceleration of a charged particle in a kicking inhomogeneous electric field and a homogeneous magnetic field [3]. The proposed model is a harmonic oscillator, experiencing a position-dependent kicking force

$$\ddot{z} + \alpha^2 z = K f(t-z) \sum_n \delta(t-n) \tag{1}$$

where $f(z) = \sin(2\pi z)$. The time-1 map of (1) can be calculated exactly. If $\alpha \neq 0$ in phase space (x, y), where $x \equiv \dot{z}/\alpha$ and $y \equiv -z$, it reads [3, 4]

$$L_\alpha = R_\alpha \circ T \tag{2}$$

where R_α is the rotation over α,

$$R_\alpha : \begin{cases} x' = \cos(\alpha)x + \sin(\alpha)y \\ y' = -\sin(\alpha)x + \cos(\alpha)y \end{cases} \tag{3}$$

$\circ$ denotes composition, and

$$T : \begin{cases} x' = x + K_0 f(y) \\ y' = y \end{cases} \tag{4}$$

where $K_0 = K/\alpha$. L_α is area preserving (and hence symplectic). Here we consider the case $\alpha \neq 0$, the case $\alpha = 0$ requires a separate treatment. In case $\alpha = 0$ the time-1 map of (1) is the standard map [1, 4, 5].

In previous studies [3, 4] it was numerically observed that if $\alpha/2\pi$ is rational, i.e. the eigenfrequency of the oscillator and the frequency of the kicks are at resonance, the stochastic web looks extremely symmetric, see e.g. figure 1. Moreover, for some of the resonances it has been shown that there is no threshold for stochastic diffusion, i.e. whenever $K_0 \neq 0$ there is stochastic diffusion [6].

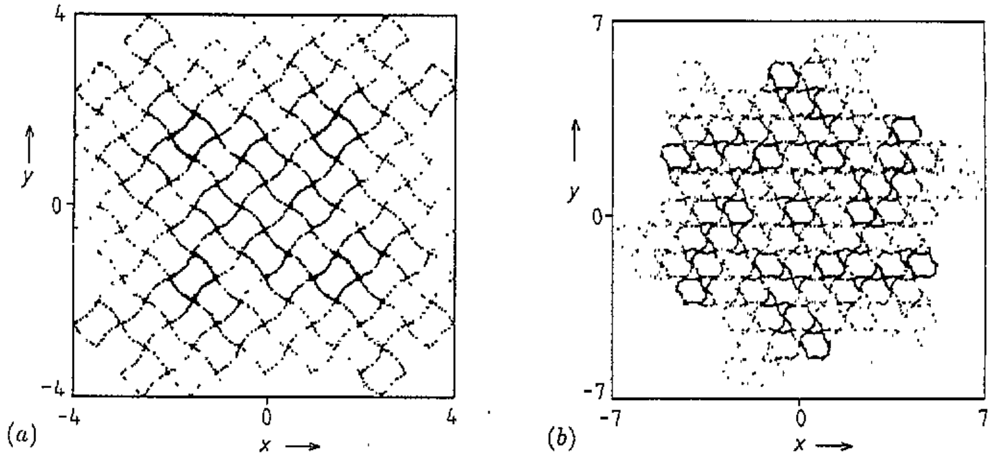


Figure 1. Part of the stochastic web generated by the map L_α for the case $f(y) = \sin(2\pi y)$, $K_0 = 0.18$ and (a) $\alpha = \pi/2$, (b) $\alpha = \pi/3$.

In this paper we will study the symmetries of stochastic webs of L_α, where f is a periodic function $f(y) = f(y + 1)$, as proposed by Hoveijn [7]. Via an analysis of the equations of motion, we will show that at resonances that satisfy a crystallographic condition, the stochastic webs possess crystallographic symmetries. Furthermore, a symmetry argument will be given to explain the fact that one finds no threshold for stochastic diffusion if $\alpha = \pm\pi/2$, f is odd and $f(y + 1/2) = -f(y)$.

2. Symmetries and reversing symmetries

In the symmetry analysis that we will perform in the next sections, we will distinguish between symmetries and reversing symmetries. We say that a map M is a symmetry of a map L if

$$M \circ L \circ M^{-1} = L. \tag{5}$$

We say that L has a reversing symmetry S, if

$$S \circ L \circ S^{-1} = L^{-1}. \tag{6}$$

Note that if the order of the reversing symmetry S, i.e. the smallest integer n such that $S^n = Id$, is odd then automatically $L^2 = Id$.

If L has a reversing symmetry then the map is called *weakly reversible* in general, and *reversible* if the reversing symmetry is an involution, i.e. its own inverse [8, 9].

The symmetries of a map, together with its reversing symmetries form a group under composition. We call this group the *reversing symmetry group* of the map. The composition of a symmetry and a reversing symmetry is a reversing symmetry and the composition of two (reversing) symmetries is a symmetry. For a more detailed discussion on the structure of reversing symmetry groups and a related decomposition property of weakly reversible maps, see [10].

It has been well recognized that the map L_α is reversible [7, 11–13]. Let M_y be the mirror in the y-axis

$$M_y : \begin{cases} x' = -x \\ y' = y \end{cases} \tag{7}$$

then we find that

$$(R_\alpha \circ M_y)^2 = (M_y \circ T)^2 = Id \tag{8}$$

and by the decomposition property of reversible maps [9] that $R_\alpha \circ M_y$ is an involutory reversing symmetry of L_α.

Moreover, if f is odd, i.e. $f(-y) = -f(y)$, then $-Id$ is a symmetry of L_α and hence $-Id \circ R_\alpha \circ M_y \equiv R_\alpha \circ M_x$ is also a reversing involution.

Up to here, we find that L_α has definite symmetry properties. However, we do not find that L_α has the translation symmetries that the stochastic webs in figure 1 seem to possess. In the next section we will show that translation symmetries are possessed by powers of L_α at resonances that satisfy a crystallographic restriction.

3. The translation group and its lattice

In this section it will be shown that if $\alpha_{p,q} = 2\pi p/q$ where p and q are integer coprimes and $q \in \{3, 4, 6\}$ that $L^q_{\alpha_{p,q}}$, i.e. the time-q map of (1), possesses translation symmetries. This is also the case if $q = 1$ and $q = 2$, but we do not consider these cases here, since for these values of q the mapping displays no chaotic motion [3, 4]. As a result of the translation symmetries, we find that there is a subgroup of the reversing symmetry group of $L^q_{\alpha_{p,q}}$ that is isomorphic to a crystallographic group.

To prove this, we first observe that any resonance $\alpha_{p,q}$, $L^q_{\alpha_{p,q}}$ can be written as

$$L^q_{\alpha_{p,q}} = (R_{\alpha_{p,q}} \circ T)^q = \prod_{k=1}^{q} T_{k \cdot \alpha_{p,q}} \tag{9}$$

where $\prod$ denotes successive composition of the maps

$$T_\beta = R_\beta \circ T \circ R_{-\beta} \tag{10}$$

where in (9) $\beta = k \cdot \alpha_{p,q}$. At this point, let us first consider the translation symmetries of T. T possesses the translation symmetries

$$U_{r,n} : \begin{cases} x' = x + r \\ y' = y + n \end{cases} \tag{11}$$

for all $r \in \mathbb{R}$ and $n \in \mathbb{Z}$.

Regarding (9) it is obvious that if there exists a translation $U_{r,n}$ that commutes with $T_{k\cdot\alpha_{p,q}}$ for all k then $L^q_{\alpha_{p,q}}$ possesses this translation symmetry. Moreover, if $U_{r,n}$ is such a translation symmetry, then it is inevitable that also $R_{k\cdot\alpha_{p,q}} \circ U_{r,n} \circ R_{-k\cdot\alpha_{p,q}}$ is a translation symmetry for any integer k. However this restricts the possible translation symmetries to the ones that satisfy $R_{\alpha_{p,q}} \circ U_{r,n} \circ R_{-\alpha_{p,q}} = U_{r',n'}$ for some $r' \in \mathbb{R}$ and $n' \in \mathbb{Z}$. It is a well known fact from crystallography [14, 15] that this can happen only if $q \in \{1, 2, 3, 4, 6\}$. In other words, a pattern on a plane with translation symmetry can possess only two-, three-, four-, and sixfold rotocentres that leave this pattern invariant.

In this way we find that $L^4_{\alpha_{p,4}}$ possesses a group of translation symmetries generated by $U_{1,0}$ and $U_{0,1}$. In case $q = 3$ or $q = 6$, $L^q_{\alpha_{p,q}}$ possesses a group of translation symmetries generated by $U_{2/\sqrt{3},0}$ and $U_{1/\sqrt{3},1}$. In both cases the translation vectors span a lattice. In case $q = 4$ the lattice is square and in case $q = 3$ or $q = 6$ the lattice is hexagonal.

Because of the translation symmetries, the phase space can be considered to be tiled with unit cells that contain equivalent dynamics with respect to $L^q_{\alpha_{p,q}}$. Hence, as in crystallography, from here on we can restrict our attention to the dynamics in one unit cell. In the same way as unit cells can be decorated with atoms in crystallography, here the unit cells can be regarded to be decorated with dynamics. The actual decoration of the unit cell depends on the function f. In case $q = 4$ the unit cell is a 1×1 square box, and in case $q = 3$ or $q = 6$ the unit cell is a diamond with perpendicular diagonals of length $2/\sqrt{3}$ and 2.

Because of the translation symmetries one could also regard the unit cell to represent the surface of a two-torus on which the dynamics takes place.

In the next section we will consider the implication of other linear symmetries that $L^q_{\alpha_{p,q}}$ may possess.

4. Crystallographic symmetries

In this section we regard the linear discrete (reversing) symmetries that $L^q_{\alpha_{p,q}}$ ($q \in \{3, 4, 6\}$) may possess. Together with the translation symmetries, these (reversing) symmetries form a crystallographic group.

In section 2 it was shown that L_α is reversible. This reversing symmetry $R_\alpha \circ M_y$ is a mirror. In figure 2 the mirrors have been drawn in the unit cells for $p/q = \frac{1}{4}$, $p/q = \frac{1}{3}$, and $p/q = \frac{1}{6}$. The unit cells have been chosen in such a way that they have (0,0) as the centre and are symmetric with respect to the reversing mirror. The arrows in the unit cell schematically indicate the implications of the (reversing) symmetries on the dynamics in the unit cell. The crystallographic groups that are generated by the translations and the reversing mirror are isomorphic for all $q \in \{3, 4, 6\}$.

For a proper classification of the crystallographic subgroups of the reversing symmetry groups, we remark that our concept of a reversing symmetry is strongly related to the concept of a colour-reversing symmetry in dichromatic colour groups [15] and the concept of spin-reversal symmetries in magnetic groups [14]. In correspondence with the literature

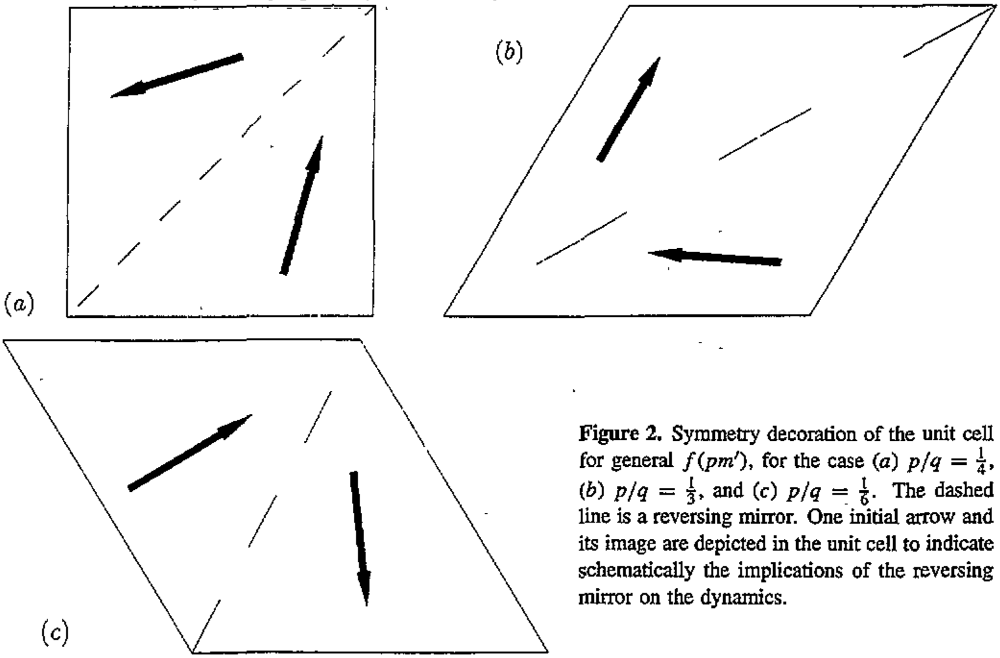


Figure 2. Symmetry decoration of the unit cell for general $f(pm')$, for the case (*a*) $p/q = \frac{1}{4}$, (*b*) $p/q = \frac{1}{3}$, and (*c*) $p/q = \frac{1}{6}$. The dashed line is a reversing mirror. One initial arrow and its image are depicted in the unit cell to indicate schematically the implications of the reversing mirror on the dynamics.

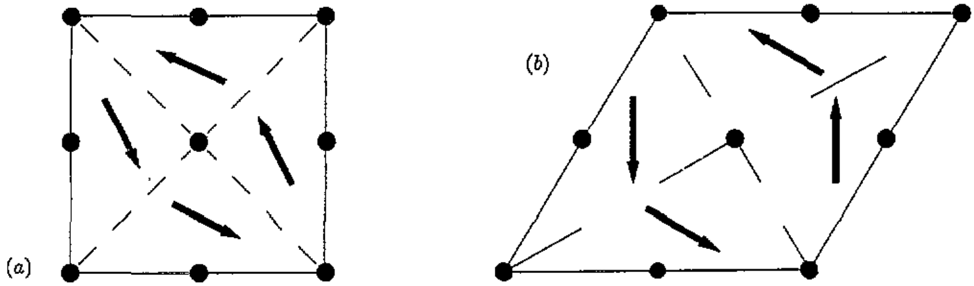
on these groups we will use the Shubnikov–Belov notation [15] for the crystallographic subgroups of the reversing symmetry group. In that notation, the group discussed above is denoted as pm' (m' denotes the reversing mirror).

If f satisfies certain constraints, more complicated crystallographic symmetries can occur. As mentioned already in section 2, if f is odd, then L_α has the symmetry $-Id$, that is a twofold rotocentre at (0,0). Hence as a direct consequence L_α possesses a second reversing mirror $R_\alpha \circ M_x$.

Moreover, if f is odd we find twofold rotocentres at $(k/2, l/2)$ if $q = 4$ and twofold rotocentres at $(k/\sqrt{3}+l\sqrt{3}/2, l/2)$ if $q = 3$ or $q = 6$ for all integers k and l. In figure 3 the reversing mirrors and rotocentres have been depicted in the unit cell for $p/q = \frac{1}{3}$, $p/q = \frac{1}{4}$ and $p/q = \frac{1}{6}$. One initial arrow and its images have been used to indicate the implications of the (reversing) symmetries on the dynamics. Again we find that the crystallographic groups for $q = 3$, $q = 4$, and $q = 6$ are isomorphic. In Shubnikov–Belov notation this group is denoted $cm'm'$.

In case $q = 4$, f is odd and $f(y + 1/2) = -f(y)$ we observe a type of reversing symmetries that we do not encounter if $q = 3$ or $q = 6$: fourfold reversing rotocentres. They are situated at $(k/2, l/2)$ where $k + l$ is odd. For a detailed derivation, see appendix A. In figure 4 the mirrors and rotocentres have been drawn in the square unit cell. Again the images of one arrow have been used to indicate the implications of the (reversing) symmetries on the dynamics. In this case the crystallographic subgroup of the reversing symmetry group is $p4'gm'$.

The crystallographic subgroup of the reversing symmetry group of the map $L^q_{\alpha_{p,q}}$ contains the apparent symmetries of its stochastic web. A stochastic web consists of a collection of chaotic orbits of the map L_α (that is of course identical to the stochastic web of the map L^k_α for any non-zero integer k). Every (reversing) symmetry in the crystallographic subgroup of the reversing symmetry group relates points of the stochastic web to points of other chaotic orbits. It is easily checked that these other chaotic orbits must be part of the stochastic web too. This is the reason why the stochastic web of the map $L_{\alpha_{p,q}}$ clearly displays all the symmetries of the crystallographic subgroup of the reversing symmetry group of $L^q_{\alpha_{p,q}}$.



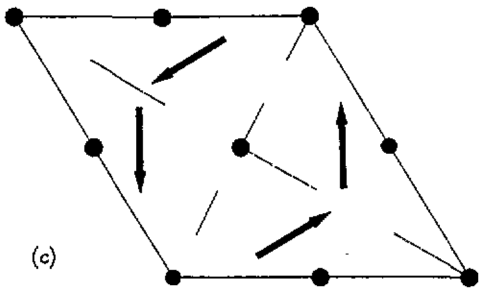


Figure 3. Symmetry decoration of the unit cell for odd $f(cm'm')$, for the case (a) $p/q = \frac{1}{4}$, (b) $p/q = \frac{1}{3}$, and (c) $p/q = \frac{1}{6}$. The dashed lines are reversing mirrors and the ●'s denote twofold rotocentres. One initial arrow and its images are depicted in the unit cell to indicate the implications of the (reversing) symmetries on the dynamics.

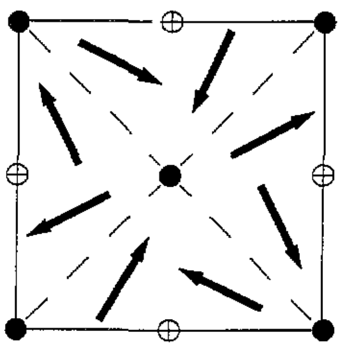


Figure 4. Symmetry decoration of the unit cell for the case where f is odd, $f(y + 1/2) = -f(y)$ and $p/q = \frac{1}{4}$ $(p4'gm')$. Dashed lines denote reversing mirrors, ●'s twofold rotocentres, and ⊕'s fourfold reversing rotocentres. One initial arrow and its images are depicted in the unit cell to indicate the implications of the (reversing) symmetries on the dynamics.

However, in the description of the symmetries of the stochastic web it is not relevant to distinguish between symmetries and reversing symmetries. Hence to find the symmetry group of the stochastic web, we have to derive from the crystallographic subgroup of the reversing symmetry group, the ordinary crystallographic group that is found by neglecting the differences between symmetries and reversing symmetries. We find that if $q \in \{3, 4, 6\}$ the stochastic webs have pm symmetry in general, and cmm symmetry if f is odd. In case $q = 4$, f is odd and $f(y + 1/2) = -f(y)$, the stochastic web has $p4gm$ symmetry. The results have been summarized in table 1.

Table 1. The crystallographic subgroup of the reversing symmetry group of $L^q_{\alpha_{p,q}}$ (RSG) and the crystallographic symmetry group of its stochastic web (SG web) for different values of q and symmetry properties of the periodic function f.

Property of f	q	RSG	SG web
—	3,4,6	pm'	pm
$f(y) = -f(-y)$	3,4,6	$cm'm'$	cmm
$f(y) = -f(-y)$ and $f(y+1/2) = -f(y)$	4	$p4'gm'$	$p4gm$

Remark that from the analysis above it follows that the stochastic webs that have been observed by Zaslavsky and co-workers have *cmm* symmetry if $q = 3$ or $q = 6$ and *p4gm* symmetry if $q = 4$, because they considered $f(y) = \sin(2\pi y)$ that is odd and satisfies $f(y + 1/2) = -f(y)$. In figure 5 the stochastic webs have been plotted in the unit cells for this particular choice of f. In combination with figure 3 and figure 4 it is easy to check the symmetry properties of the web.

5. Approximate Hamilton systems

In [4, 5] it is suggested that the symmetry of the stochastic web of the map $L_{\alpha_{p,q}}$ with $f(y) = \sin(2\pi y)$, is related to the symmetry of the Hamiltonian

$$H_{\alpha_{p,q}}(\boldsymbol{x}) = \sum_{k=1}^{q} g(\boldsymbol{x} \cdot \hat{\boldsymbol{e}}_k) \tag{12}$$

where $\hat{\boldsymbol{e}}_k = 2\pi(-\sin(k\alpha_{p,q}), \cos(k\alpha_{p,q}))$ and $(\mathrm{d}/\mathrm{d}y)g(y) = -f(y)$. In fact, Hoveijn [7] has shown that the map $L^q_{\alpha_{p,q}}$ approximates the time-K_0 map of $H_{\alpha_{p,q}}$ with error $\mathcal{O}(K_0^2)$. Hence, the error of this approximation is small if K_0 is small. Note however that if q is even, this approximation suppresses the even part of f: if $f = f_{\mathrm{even}} + f_{\mathrm{odd}}$, then the contribution of f_{even} to $H_{\alpha_{p,q}}$ is zero. Hence, if q is even then this approximation should be used only in case f is odd.

The approximate Hamilton system can be used to obtain some insight into the structure of the stochastic web for small K_0.

The type of chaos that creates the stochastic web stems from transversal intersections of the stable manifold of one hyperbolic fixed point and the unstable manifold of another hyperbolic fixed point; so-called heteroclinic points. These intersections create a type of saddle connection along which the chaotic motion proceeds. The microscopic structure of these type of saddle connections is very complicated (see e.g. figure 5).

The web that is formed by the saddle connections in the approximate Hamilton system can serve as a first approximation of the stochastic web. This first approximation to the stochastic web is called the web's skeleton, following [4, 5].

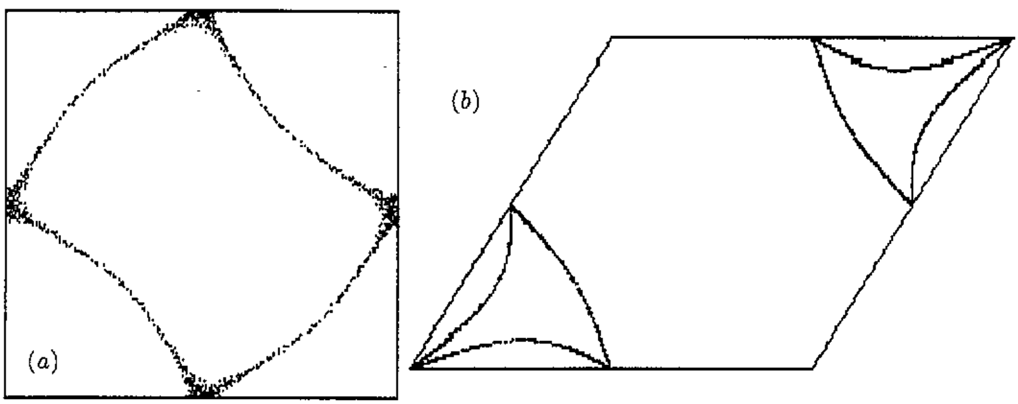
At this stage it is of interest to study the symmetries of the approximate Hamilton systems and to see to what extent they display the symmetries of the map $L_{\alpha_{p,q}}$ for $q \in \{3, 4, 6\}$.

The linear symmetries and reversing symmetries of the approximate Hamilton system can be read from the Hamiltonian immediately (see appendix B). We find in case $q \in \{3, 4, 6\}$, that $H_{\alpha_{p,q}}$ possesses the same translation symmetries as the maps $L^q_{\alpha_{p,q}}$. However, the approximate Hamilton system does not share all its other linear (reversing) symmetries with $L^q_{\alpha_{p,q}}$. The crystallographic subgroups of the reversing symmetry groups of the approximate Hamilton systems and the consequences of that for the symmetries of the web's skeleton, found by ignoring differences between symmetries and reversing symmetries, are summarized in table 2. Comparing table 2 with table 1 we find that the crystallographic subgroups of the reversing symmetry groups of $L^q_{\alpha_{p,q}}$ that were found in the previous section, are subgroups of the crystallographic groups that we find in the analysis of the approximate Hamilton systems. In other words, the neglected error in the Hamiltonian approximation breaks some of the symmetries of the approximate Hamilton system.

Table 2. The crystallographic subgroup of the reversing symmetry group of the approximate Hamilton system (RSG) and the crystallographic symmetry group of the web's skeleton (SG skeleton) for different values of q and symmetry properties of the periodic function f.

Property of f	q	RSG	SG skeleton
—	3	$p3m'$	$p3m$
$f(y)=-f(-y)$	3,6	$p6m'm'$	$p6mm$
$f(y)=-f(-y)$	4	$p4m'm'$	$p4mm$
$f(y)=-f(-y)$ and $f(y+1/2)=-f(y)$	4	$p'4gm$	$p4mm$

The symmetry breaking can easily be verified by comparing figures 2–4 with figure 6†. In particular, in the case that $q=3$ and there is no restriction on f, the threefold rotocentre that appears in the approximate Hamilton system is absent in the reversing symmetry group of $L^q_{\alpha_{p,q}}$. In case f is odd, the four- and sixfold rotocentres in the approximate Hamilton systems become twofold rotocentres. In case $f(y+1/2)=-f(y)$ and $q=4$ the twofold reversing rotocentres disappear and at the same time the mirrors between the fourfold reversing rotocentres are broken.



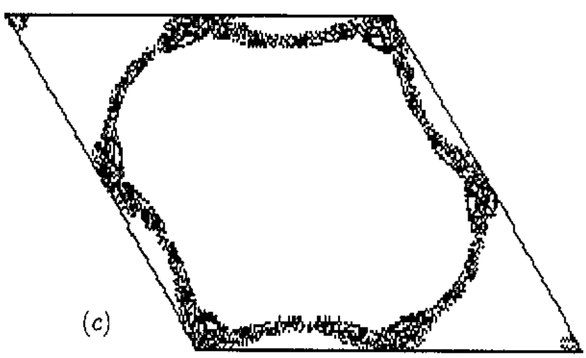


Figure 5. Part of the stochastic web generated by $L_{\alpha_{p,q}}$ with $K_0 = 0.18$ in the unit cell for (*a*) $p/q=\frac{1}{4}$, (*b*) $p/q=\frac{1}{3}$, and (*c*) $p/q=\frac{1}{6}$.

In the next section the last type of symmetry breaking will be used to show that generically in case $q=4$, f is odd, and $f(y+1/2)=-f(y)$ there is no threshold for stochastic diffusion.

† In figure 6 for reasons of clarity we have chosen not to display arrows to indicate the implications of the (reversing) symmetries on the dynamics.

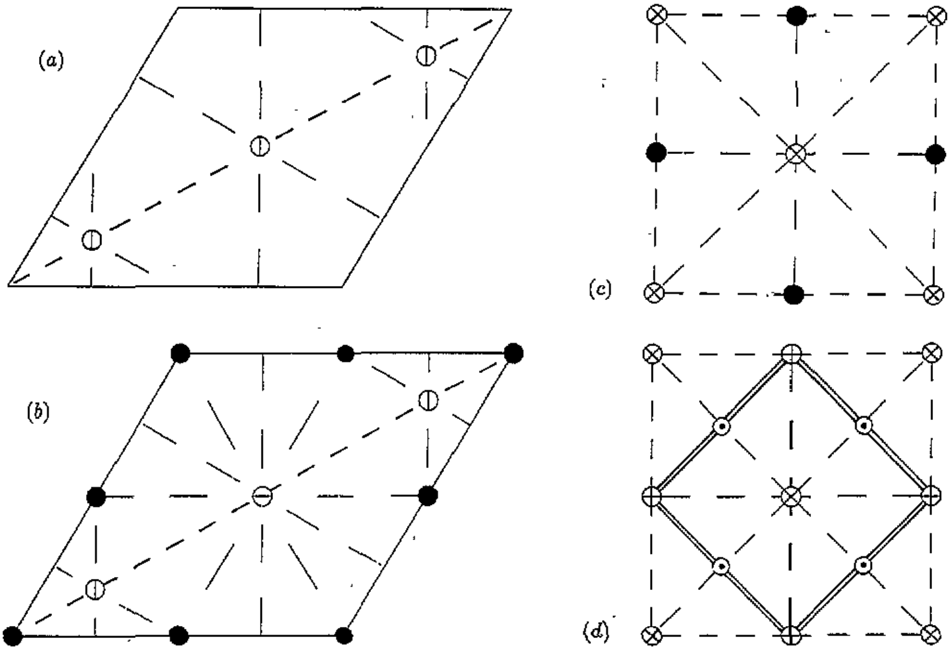


Figure 6. Decoration of the unit cells of the dynamics generated by $H_{\alpha_{p,q}}$ for the case (*a*) $q = 3$ ($p3m'$), (*b*) $q = 3, 6$ ($p6m'm'$), (*c*) $q = 4$ ($p4m'm'$), (*d*) $q = 4$ ($p'4gm$). Dashed lines denote reversing mirrors, ⊕'s fourfold reversing rotocentres, ⊙'s twofold reversing rotocentres, ⊖'s sixfold rotocentres, ⊗'s fourfold rotocentres, ⦶'s threefold rotocentres, ●'s twofold rotocentres, and double lines denote mirrors.

6. On the threshold for stochastic diffusion

In the standard map it is well known that there is a threshold for stochastic diffusion that is related to the break-up of the last KAM-torus [1]. However, in the case of the maps that were considered by Zaslavsky and co-workers it was shown that there is no threshold for stochastic diffusion for the map $L_{\alpha_{p,q}}$ at the resonances $\alpha_{p,q}$ with $q \in \{3, 4, 6\}$ in case $f(y) = \sin(2\pi y)$ [6]. At resonances $\alpha_{p,q}$ with $q \notin \{1, 2, 3, 4, 6\}$ there is less certainty about the threshold, but if there is one it is believed to be extremely small.

If the plane is tiled with finite-size unit cells, a stochastic web occurs if the chaotic saddle connections connect the sides of the unit cell to form a two-dimensional web. Then the chaotic motion spreads over the entire phase space along the saddle connections that form the stochastic web. This has been observed to occur immediately for every non-zero value of K_0 in case $f(y) = \sin(2\pi y)$ and $q \in \{3, 4, 6\}$.

One way of understanding the fact that there is no threshold for stochastic diffusion is to regard the saddle connections in the approximate Hamilton systems. If they form a web (the web's skeleton) then this web may survive for small values of K_0 if one takes into account the (small) errors that were made in the Hamiltonian approximation [4, 7].

Now raises the question whether we can predict from the function f whether we will find a threshold for stochastic diffusion, or not. Regarding the case $q = 4$, f is odd and $f(y + 1/2) = -f(y)$, by a symmetry argument it will be shown that generically one will not find a threshold for stochastic diffusion.

Consider the (reversing) symmetries as indicated in the unit cell in figure 6(*d*). The fourfold reversing rotocentres play an important role in the stochastic web. If a dynamical system on the plane possesses a fourfold reversing rotocentre, then

(i) The fourfold reversing rotocentre point is a fixed point.
(ii) The character of this fixed point is generically hyperbolic. (The parabolic case is the marginal exception.)

The first observation follows directly from the fact that any fourfold reversing rotocentre is also a twofold rotocentre. It is easy to show that any twofold rotocentre point is a fixed point. The second observation is related to the fact that any 2×2 matrix with real entries that is inverted after conjugation with $R_{\pi/2}$ has a determinant equal to one, and two real eigenvalues.

Now, consider the generic situation that the fourfold reversing rotocentres are occupied by hyperbolic fixed points. *In the Hamiltonian approximation two of such fourfold reversing* rotocentres are connected via a mirror (double lines in figure 6(d)). This implies immediately that this mirror serves as a saddle connection, i.e. the mirror is the stable manifold of one of the hyperbolic points and the unstable manifold of the other. Hence if $q = 4$, f is odd and $f(y + 1/2) = -f(y)$ the approximate Hamilton system possesses a web of saddle connections: the web's skeleton of the stochastic web of $L_{\alpha_{p,q}}$†.

Regarding the (reversing) symmetries of $L^q_{\alpha_{p,q}}$ in the case that $q = 4$, f is odd, and $f(y + 1/2) = -f(y)$ (figure 4), we observe that the fourfold reversing rotocentres are still there, so generically these points are hyperbolic fixed points. The reversing mirrors also survive, but the ordinary mirrors, which served as the web's skeleton, have disappeared. Hence, generically the smooth saddle connections break up. Nevertheless the stable and unstable manifolds that become separated will still intersect a reversing mirror. Moreover, if a stable or unstable manifold intersects a reversing mirror, then it follows immediately that this point is a heteroclinic point, i.e. an intersection point of a stable manifold of one hyperbolic point and the unstable manifold of another hyperbolic point. Thus, although the smooth saddle connection generically disappears, some heteroclinic points will survive. These heteroclinic points give rise to a chaotic type of saddle connections. Since these chaotic saddle connections form a web, we find that generically there is no threshold for stochastic diffusion, i.e. for every non-zero value of K_0, stochastic diffusion is to be expected.

It would be tempting to conjecture that the above conclusion can be drawn without using the approximate Hamiltonian, using only the (reversing) symmetries of the map, by showing that generically it is inevitable that the stable and unstable manifolds of the hyperbolic fixed points on the fourfold reversing rotocentres intersect reversing mirrors. However, more work is needed to gain the insight that is needed to prove such a conjecture.

7. Concluding remarks

In this paper it has been shown that crystallographic symmetries of stochastic webs can be understood entirely from an analysis of the equations of motion.

One should be careful using the approximate Hamilton system in explaining symmetry properties of the stochastic web. This notion is important in relation to questions concerning the symmetry properties of the stochastic web in case $q \notin \{1, 2, 3, 4, 6\}$. In [4,5] *it is conjectured that since the approximate Hamilton system possesses quasi-periodic* symmetries, the stochastic web will too. From the present analysis it follows that one

† For a picture that illustrates this fact, see e.g. [4] p 130.

should be careful with such conjectures since it is to be expected that the stochastic web breaks some of the symmetries of the approximate Hamilton system.

However, this problem deserves a discussion on its own and this is beyond the scope of the present paper. In analogy with methods that are used in quasi-crystallography [16], one may think of proving the existence of quasi-crystallographic symmetries by embedding the dynamical system in a higher-dimensional phase space in which it possesses translation symmetries.

The (reversing) symmetries of (powers of) a map, of course, do not just affect the chaotic motion. Also, regular orbits may turn out to be symmetric because of (reversing) symmetries of (powers of) the map. For a discussion on the symmetries of the regular motion in relation to the symmetries of the chaotic motion, see e.g. Chossat and Golubitsky [17] and Kimball and Dumas [18].

A general discussion on dynamical systems that possess (reversing) symmetries only if they are considered on a proper time scale, e.g. powers of a map possess (reversing) symmetries that the map itself does not possess, is the subject of a future publication [19].

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Appendix A. A fourfold reversing rotocentre for the case $q = 4$, f is odd and $f(y+1/2) = -f(y)$

In this appendix it is shown that for the case $q = 4$, f is odd, and $f(y+1/2) = -f(y)$, $L^q_{\alpha_{p,q}}$ possesses a fourfold reversing rotocentre at (1/2, 0).

If $q = 4$ and f is odd we can write

$$L^4_{p\pi/2} = (T_{p\pi/2} \circ T)^2 \, (p = \pm 1) \tag{13}$$

as explained in section 3. If we now perform the translation $(x, y) \mapsto (x + 1/2, y)$, we find, if $f(y + 1/2) = -f(y)$, that $L^4_{p\pi/2} \mapsto \tilde{L}^4_{p\pi/2}$, where

$$\tilde{L}^4_{p\pi/2} = (T^{-1}_{p\pi/2} \circ T)^2. \tag{14}$$

From (14) it is easily checked that

$$R_{\pi/2} \circ \tilde{L}^4_{p\pi/2} \circ R_{-\pi/2} = (T^{-1} \circ T_{p\pi/2})^2 = (\tilde{L}^4_{p\pi/2})^{-1}. \tag{15}$$

Hence (1/2, 0) is a fourfold reversing rotocentre of $L^4_{\alpha_{p,q}}$. From the other (reversing) symmetries of $L^4_{\alpha_{p,q}}$, it follows directly that there are fourfold reversing rotocentres at all points $(k/2, l/2)$ with k and l integers and $k + l$ odd.

Appendix B. Hamilton systems with linear (reversing) symmetries

In this appendix the property of Hamilton systems of having linear (reversing) symmetries, is related to a symmetry property of the Hamiltonian.

Proposition. Consider a Hamilton system with Hamiltonian $H(x)$ on the phase space $x = (q, p) \in \mathbb{R}^n \times \mathbb{R}^n$, and a linear transformation U that can be written as a 2×2 matrix on (q, p). Then

U is a symmetry $\Leftrightarrow H(Ux) = \det(U) \cdot H(x)$

U is a reversing symmetry $\Leftrightarrow H(Ux) = -\det(U) \cdot H(x)$.

Moreover, if T is a translation, then

T is a symmetry $\Leftrightarrow H(Tx) = H(x)$,

T is a reversing symmetry $\Leftrightarrow H(Tx) = -H(x)$.

Proof. In a Hamilton system the equations of motion are given by

$$\dot{x} = J\nabla_x H(x) \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{16}$$

If $x = Ux'$ then $\nabla_{x'} = U^{\mathrm{T}}\nabla_x$ and with (16) it follows that

$$U^T \circ J^{-1} \circ U\dot{x}' = \nabla_{x'} H(Ux'). \tag{17}$$

Moreover

$$U^T \circ J^{-1} \circ U = \det(U) \cdot J^{-1} \tag{18}$$

and hence

$$\dot{x}' = J\nabla_{x'} \frac{H(Ux')}{\det(U)}. \tag{19}$$

U preserves the equations of motion if $H(Ux) = \det(U) \cdot H(x)$ and U inverts the equations of motion if $H(Ux) = -\det(U) \cdot H(x)$. □

The proof for the translations is easy, and therefore omitted.

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